A CLASSICAL POINT CHARGE SPONTANEOUSLY FREE OF SINGULAR SELF ACTION

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It is shown that within classical electromagnetism there is at least one, and very likely many other, particle behaviors that are innately free of singular self-action, provided the bare charge is massless. A likely outcome is that classical electromagnetism supplies its own remedy for rendering the traditionally infinite Coulomb self-energy of a point particle finite, without recourse to form-factors or mass-renormalization. The remedial motion discussed here has some interesting qualities suggestive of a correspondence between massless classical electromagnetism and modern particle physics.

1. Introduction

1.1. *Background*

The classical point charge has an infinite self-energy. In Gaussian units the electric field energy density of an electron at the origin is $e^2/2r^4$, the integration of which diverges if taken down to the origin. The energy can be rendered finite by asserting a finite sized structure for the particle of the order of the classical electron radius [1, 2]. But no such structure at the predicted scale has been observed. The quantum-theoretical electron suffers from the same problem for the same reason that the particle is structureless. It suffers additionally divergent behavior in the presence of the second-quantized vacuum field (which itself has an infinite energy density). The traditional remedy is to absorb the Coulomb-type self-energy into the definition of the mass. The device amounts to asserting a negative infinite 'bare' mass, chosen so that the two contributions sum to give the finite positive observed mass. Though mathematically satisfactory, for some, the technique is philosophically troubling.

In this article we will show that within classical electromagnetism (CEM), there exist particle motions for which the total self-action is finite, even if self-action is allowed. The novel ingredients are that the bare charge is massless, and that the interaction between particles is time-symmetric. The former is discussed in another article in this collection [3]. The property of time-symmetry is probably usually associated with the direct-action versions of EM after Schwarzschild [4], Tetrode [5], and Fokker [6], and by Stueckelberg [7, 8]. Being intrinsically time-symmetric, direct-action CEM calls for some explanation of the appearance of (time-asymmetric) retarded radiation. The Wheeler and Feynman absorber theory [9, 10] suggests that relatively cold distant absorbers on the future light cone play the role of thermal sink traditionally attributed to the vacuum. The viability of the idea, however, depends on accord with Cosmological observations [11], which issue will not be persued here.

This effort is motivated by the following observation: The traditional retarded electric field of a point particle at the origin with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$, is [2, 12]

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\left(1-v^2\right)(\mathbf{x}-x\mathbf{v}) + \mathbf{x}\times((\mathbf{x}-x\mathbf{v})\times\mathbf{a})}{(x-\mathbf{x}\cdot\mathbf{v})^3}\right]_{ret}. \tag{1}$$

(Here and throughout will be adopted Gaussian units with $c = 1$.) The first and second terms are commonly called the velocity and acceleration fields respectively. One notices at first that the singular nature of the Coulomb self-energy is a result of the denominator $(x-\mathbf{x}\cdot\mathbf{v})^3$ in the velocity field. (The acceleration field diverges like $1/x$ as $x \to 0$ and therefore the square integrates to a finite value.) The motivation for this effort comes from the observation that if $v > 1$, $(x-\mathbf{x}\cdot\mathbf{v})^3$ can also go to zero at other x. Since it has already been demonstrated [3] that superluminal motion is a consequence of innate masslessness, the conjecture is that a massless bare charge can move so as to intersect its own light cone in such a manner as to cancel the 'Coulomb singularity'. That investigation into the massless condition led to the conclusion that self-action cannot be denied after all, even in direct-action EM where apparently one can strike it out by hand. If instead self-action is permitted, but then precisely cancelled by suitable motion, an innately massless charge becomes a viable possibility.

The above discussion based upon Eq. (1) is a preliminary sketch. The following will use advanced and retarded fields, and will be concerned with cancellation of the total self-force, not just the contribution from the electric field. The condition of time-symmetric interaction is assumed to apply at some primitive, effectively zero-Kelvin, level of description, certainly prior to the emergence of radiation and radiation reaction, and possibly also limited to a level of description prior to the emergence of mass-energy associated with individual particles.

1.2. *Outline of the method*

The time-symmetric electromagnetic direct action of a single charged particle is [12]

$$I = -e^2 \int d\lambda \int d\kappa x^{\alpha}(\lambda)\, x_{\alpha}(\kappa)\, \delta\left((x(\lambda) - x(\kappa))^2\right), \tag{2}$$

where λ and κ are any ordinal parameterizations of the trajectory. Since the bare charge is assumed massless, the following will assume that this is the only action. As a result, the object in question can move at superluminal speeds. As a consequence it will turn out that in general $x(\lambda)$ cannot be identified with the 4-location of a particle, nor will the number of particles be limited to one, even though, at any λ-time, there remains only one $x(\lambda)$ 'event' (see the discussion in section 3.1). In order therefore to distinguish $x(\lambda)$ from a particle position, here it will be referred to simply as a trajectory, without necessarily implying the trajectory of anything in particular.

The traditional course is to minimize Eq. (2) by variation of $x(\lambda)$ subject to fixed end-points. This leads to an Euler equation that is a 2^{nd} order differential-difference equation with infinitely many difference terms. The goal here is much more modest than the solution of that system. The goal is only to show that there exists some motion for which the total energy is finite or zero. Strictly, even that goal will not be attempted here. Instead we will look for an expression for the trajectory $x(\lambda)$ as a function of λ (hereafter to be loosely termed 'motion') that renders the total action finite over a finite time interval. The justification for this approach is given in section 3.2.

In the following we will consider a simple case of circular motion at constant speed. In the static frame of observation (laboratory frame) there will be no time reversals, and so it will be safe to use the laboratory coordinate t to parameterize the trajectory, whereupon Eq. (2) can be written

$$I = -e^2 \int dt \int dt' \left(1 - \mathbf{v}(t) \cdot \mathbf{v}(t')\right) \delta\left((t - t')^2 - (\mathbf{x}(t) - \mathbf{x}(t'))^2\right). \tag{3}$$

The aim is to see if there exists a motion such that the singular behavior in Eq. (2) near $t = t'$, can be cancelled by other contributions. To that end it will be useful to consider a related action that is not singular at $t = t'$ but which becomes singular in some limit. Accordingly we will work with

$$I(\Delta) = -e^2 \int dt \int dt' \left(1 - \mathbf{v}(t) \cdot \mathbf{v}(t')\right) \delta\left((t - t')^2 - (\mathbf{x}(t) - \mathbf{x}(t'))^2 + \Delta^2\right) \tag{4}$$

and later let $\Delta \to 0$. This device is similar to that employed in a regularized direct-action CEM [13], though there the regularization parameter was of opposite sign and remained finite). Circular motion in direct action CEM has been studied by

Schild [14, 15], and the two body problem in general (i.e. not specifically circular motion) has been studied by many others. But in those works self-action is always excluded or renormalized, and the charges are attributed with intrinsic mass, and so are limited to sub-luminal speeds. Consequently two - independent - charged trajectories are involved, rather than the single self-interacting trajectory described here.

2. Implementation

2.1. *Action of the trial solution*

The proposed trial solution that renders the action finite is just circular motion at arbitrary radius a and angular frequency ω. For simplicity the motion is confined to the x, y plane with the center of the circle at the origin, so that in a Cartesian basis the position is

$$\mathbf{x}(t) = a\left(\cos\omega t, \sin\omega t, 0\right) \tag{5}$$

and the velocity is

$$\mathbf{v}(t) = \omega a\left(-\sin\omega t, \cos\omega t, 0\right). \tag{6}$$

These give

$$\begin{aligned}\left(\mathbf{x}\left(t\right) - \mathbf{x}\left(t'\right)\right)^2 &= a^2\left(\cos^2\omega t + \cos^2\omega t' + \sin^2\omega t + \sin^2\omega t' - 2\cos\left(\omega\left(t - t'\right)\right)\right) \\ &= 4a^2\sin^2\left(\omega\left(t - t'\right)/2\right) \end{aligned} \tag{7}$$

and

$$\mathbf{v}\left(t\right)\cdot\mathbf{v}\left(t'\right) = \omega^2 a^2\cos\left(\omega\left(t - t'\right)\right) \tag{8}$$

respectively. Eqs. (7) and (8) in (4) give that the action is

$$I = -e^2\int dt\int dt' A\;\delta\Big(\left(t - t'\right)^2 - 4a^2\sin^2\left(\omega\left(t - t'\right)/2\right) + \Delta^2\Big)\;. \tag{9}$$

$$\text{with}\quad A = \left(1 - \omega^2 a^2\cos\left(\omega\left(t - t'\right)\right)\right)$$

Making the substitution, $u = \omega\left(t' - t\right)/2$ and identifying $v = a\omega$ as the speed of the trajectory, Eq.(9) is

$$I = -\frac{e^2\omega}{2}\int dt\int du\left(1 - v^2\cos 2u\right)\delta\left(u^2 - v^2\sin^2 u + \varepsilon^2\right) \tag{10}$$

where $\varepsilon = \omega\Delta/2$.

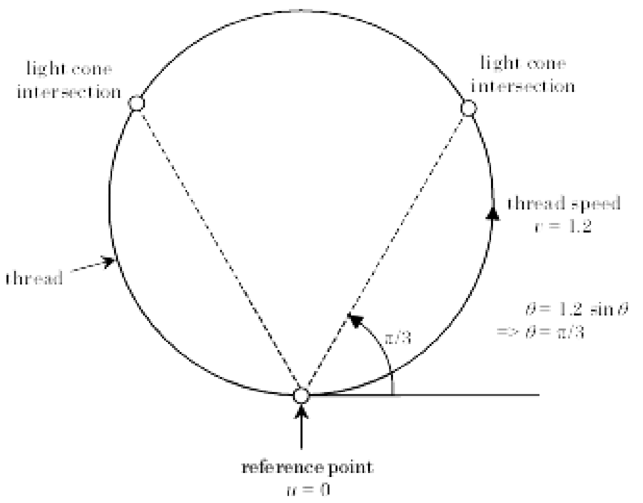

Figure 1: Superluminal circular motion.

2.2. *Roots of the delta-function*

Carrying out the integration over u in Eq. (10) gives

$$I = -\frac{e^2\omega}{2}\int dt \sum_n \frac{1 - v^2\cos 2u_n}{|2u_n - v^2\sin 2u_n|} \tag{11}$$

where n indexes the roots that are the solutions of

$$u_n^2 - v^2\sin^2 u_n + \varepsilon^2 = 0 \Rightarrow 2u_n^2 + v^2(\cos 2u_n - 1) + 2\varepsilon^2 = 0. \tag{12}$$

These roots pick out the times, for a given speed, that the trajectory intersects its own light cone. Fig. 1 shows just one pair of intersections for the speed $v = 1.2$. The number of roots depends on the magnitude of the speed, with higher speeds providing more opportunities for intersection of the light cone. For example, for speeds higher than 7.8 there are at least 4 intersection points. More generally, a new pair of intersection points is born as u is increased through each root of $\tan u = u$, these being the points at which the function $\sin x/x$ has a maximum or minimum, and therefore the points at which $\sin^2 x/x^2$ is a maximum.

Eq. (12) is even in u_n, and therefore if $u_n = k$ is a solution, then $u_n = -k$ is also a solution. Let us denote the latter by u_{-n}, and reserve the index $n = 0$ for the root u_0 that is zero when $\varepsilon = 0$, this being the traditionally troublesome infinite contribution to the self-action. With this, and noting that I is invariant to the sign of the root, Eq. (11) can be written

$$I = \frac{e^2\omega}{2}\int dt \left(K_0 + 2\sum_{n=1} K_n\right) \tag{13}$$

where the K_n are the dimensionless numbers

$$K_n = \frac{v^2 \cos 2u_n - 1}{\left|v^2 \sin 2u_n - 2u_n\right|} . \tag{14}$$

To solve for the roots of Eq. (12) let us expand the square of the velocity and the solutions u_n as power series in ε:

$$u_n = \sum_{k=0} u_{n,k}\varepsilon^k , \quad v^2 = \sum_{k=0} v^2_k \varepsilon^k \tag{15}$$

(note that the velocity is fixed with respect to each root). We now require that the action per unit time remain finite as $\varepsilon \to 0$ when these series, subject to the condition Eq. (12), are inserted into Eq. (11). Putting them first into Eq. (12) gives

$$2\left(\sum_{k=0} u_{n,k}\varepsilon^k\right)^2 + \left(\sum_{j=0} v^2_j \varepsilon^j\right)\left(\cos\left(2\sum_{k=0} u_{n,k}\varepsilon^k\right) - 1\right) + 2\varepsilon^2 = 0 . \tag{16}$$

Using that ε is small, and writing $(c_n, s_n) \equiv (\cos 2u_{n,0}, \sin 2u_{n,0})$, one has

$$\begin{aligned} \cos 2u_n &= c_n \cos\left(2u_{n,1}\varepsilon\right) - s_n \sin\left(2u_{n,1}\varepsilon + 2u_{n,2}\varepsilon^2\right) + O\left(\varepsilon^3\right) \\ &= c_n\left(1 - 2u^2_{n,1}\varepsilon^2\right) - 2s_n\left(u_{n,1}\varepsilon + u_{n,2}\varepsilon^2\right) + O\left(\varepsilon^3\right) \\ &= c_n - 2u_{n,1}s_n\varepsilon - 2\left(u_{n,2}s_n + u^2_{n,1}c_n\right)\varepsilon^2 + O(\varepsilon^3) . \end{aligned} \tag{17}$$

With this, and exchanging $(c_n, s_n) \to (s_n, -c_n)$ to obtain the corresponding result for $\sin 2u_n$, one has

$$\begin{aligned} v^2 \cos 2u_n &= v_0^2 c_n + \left(v_1^2 c_n - 2v_0^2 u_{n,1} s_n\right)\varepsilon \\ &+ \left(v_2^2 c_n - 2v_1^2 u_{n,1} s_n - 2v_0^2\left(u_{n,2}s_n + u^2_{n,1}c_n\right)\right)\varepsilon^2 + O\left(\varepsilon^3\right) \\ v^2 \sin 2u_n &= v_0^2 s_n + \left(v_1^2 s_n + 2v_0^2 u_{n,1} c_n\right)\varepsilon \\ &+ \left(v_2^2 s_n + 2v_1^2 u_{n,1} c_n + 2v_0^2\left(u_{n,2}c_n - u^2_{n,1}s_n\right)\right)\varepsilon^2 + O\left(\varepsilon^3\right) . \end{aligned} \tag{18}$$

Putting Eq. (17) in Eq. (16) and collecting terms gives

$$\begin{aligned} 2u^2_{n,0} + v^2_0\left(c_n - 1\right) &= 0 \\ 4u_{n,0}u_{n,1} + v^2_1\left(c_n - 1\right) - 2v^2_0 u_{n,1}s_n &= 0 \\ 2 + 4u_{n,0}u_{n,2} + 2u^2_{n,1} + v^2_2\left(c_n - 1\right) - 2v^2_1 u_{n,1}s_n - 2v^2_0\left(u_{n,2}s_n + u^2_{n,1}c_n\right) &= 0 . \end{aligned} \tag{19}$$

Corresponding to these, the K_n in Eq. (14) can also be expanded. Putting Eq. (18) in Eq. (14) gives

$$K_n = \frac{\left(\begin{array}{c} v^2_0 c_n - 1 + \left(v^2_1 c_n - 2v^2_0 u_{n,1}s_n\right)\varepsilon \\ + \left(v^2_2 c_n - 2v^2_1 u_{n,1}s_n - 2v^2_0\left(u_{n,2}s_n + u^2_{n,1}c_n\right)\right)\varepsilon^2 \end{array}\right)}{\left|\begin{array}{c} v^2_0 s_n - 2u_{n,0} + \left(v^2_1 s_n + 2v^2_0 u_{n,1}c_n - 2u_{n,1}\right)\varepsilon \\ + \left(v^2_2 s_n + 2v^2_1 u_{n,1}c_n + 2v^2_0\left(u_{n,2}c_n - u^2_{n,1}s_n\right) - 2u_{n,2}\right)\varepsilon^2 \end{array}\right|} + O\left(\varepsilon\right) . \tag{20}$$

2.3. *Instantaneous root*

The $n = 0$ root is defined to be zero at $\varepsilon = 0$, so $u_{0,0} = 0$ and $(c_0, s_0) = (1, 0)$. Then the first and second of Eq. (19) are already zero, and the third gives

$$1 + u_{0,1}^2 \left(1 - v^2_0\right) = 0 \Rightarrow u_{0,1} = \frac{1}{\sqrt{v^2_0 - 1}} \tag{21}$$

where, recalling the convention adopted above, $u_{0,1}$ is positive, and $u_{0,-1} = -u_{0,1}$. Near $u = 0$ therefore, there are two roots of Eq. (12):

$$u_0^+ = u_{0,0} + u_{0,1}\varepsilon + O\left(\varepsilon^2\right) = \frac{\varepsilon}{\sqrt{v^2_0 - 1}} + O\left(\varepsilon^2\right) \tag{22}$$

and

$$u_0^- = -\left(u_{0,0} + u_{0,1}\varepsilon + O\left(\varepsilon^2\right)\right) = -\frac{\varepsilon}{\sqrt{v^2_0 - 1}} + O\left(\varepsilon^2\right). \tag{23}$$

Evidently $n = 0$ is a double root. Since K is insensitive to the sign of u, K_0 in Eq. (13) can be replaced by

$$K_0 = K_0^+ + K_0^- = 2K_0^+ \tag{24}$$

so that the action Eq. (13) can be written

$$I = e^2\omega \int dt \left(K_0^+ + \sum_{n=1} K_n\right). \tag{25}$$

With these, Eq. (20) at $n = 0$ is

$$\begin{aligned} K_0^+ &= \frac{v^2_0 - 1 + v^2_1\varepsilon + \left(v^2_2 - 2v^2_0 u^2_{0,1}\right)\varepsilon^2}{|\varepsilon|\left|2v^2_0 u_{0,1} - 2u_{0,1} + \left(2v^2_1 u_{0,1} + 2v^2_0 u_{0,2} - 2u_{0,2}\right)\varepsilon\right|} + O\left(\varepsilon\right) \\ &= \frac{v^2_0 - 1}{2\left|\varepsilon\right| u_{0,1}\left|v^2_0 - 1\right|} + O\left(\varepsilon^0\right) \\ &= \frac{1}{2\left|\varepsilon\right| u_{0,1}} + O\left(\varepsilon^0\right) \end{aligned} \tag{26}$$

where the last step follows because $v^2_0 \geq 1$. The important result is that the action due to the root $n = 0$ diverges like $1/\varepsilon$ as $\varepsilon \to 0_+$. This contribution to the action gives rise to the traditionally infinite self-energy, which, in the frame in which the particle is stationary, is the self-energy of the electrostatic field. The aim then is to render the action finite by other contributions that cancel this divergence so that action in Eq. (25) is finite in that limit.

2.4. *Distant roots generating a singular contribution to the action*

The problem now is to constrain the contributions from the remaining K_n so that they diverge like $1/\varepsilon$ as $\varepsilon \to 0_+$ with the correct coefficient so as to cancel K_0 at that order:

$$\lim_{\varepsilon \to 0_+} \left(\varepsilon \left(K_0^+ + \sum_{n=1} K_n \right) \right) = 0. \tag{27}$$

In order to achieve this, it is clear that the constant term in the denominator of K_n, i.e. the term in the denominator that remains when $\varepsilon = 0$, must be zero. From Eq. (20) it is observed that this requires that

$$2u_{n,0} = v^2_0 \sin 2u_{n,0}, \tag{28}$$

in which case K_n becomes

$$K_n = \frac{v^2_0 c_n - 1}{\varepsilon \left| v^2_1 s_n + 2v^2_0 u_{n,1} c_n - 2u_{n,1} \right|} + O\left(\varepsilon^0\right) \tag{29}$$

now having the desired singular behavior as $\varepsilon \to 0$. Eq. (28) combined with first of Eq. (19) gives either $u_{n,0} = 0$, which is discounted because it is already accounted for in the root $n = 0$, or

$$u_{n,0} \sin 2u_{n,0} + \cos 2u_{n,0} = 1 \Rightarrow 2u_{n,0} \sin u_{n,0} \cos u_{n,0} = 2\sin^2 u_{n,0}. \tag{30}$$

The trivial solutions

$$u_{n,0} = 0 \text{ or } \sin u_{n,0} = 0 \Rightarrow u_{n,0} = 0 \text{ or } \pi \Rightarrow t = t' \text{ or } \omega(t - t') = 2\pi \tag{31}$$

are again discounted. The remaining solution is

$$\tan u_{n,0} = u_{n,0}, \quad u_{n,0} \neq 0,\ n \neq 0, \tag{32}$$

which, therefore, is the condition for the desired singular behavior from the $n \neq 0$ roots. The roots can be read off from the intersection points of the two curves $y = \tan x$ and $y = x$, shown in Fig. 2.

Clearly there are a doubly-infinite number of solutions. It might appear then that the index n in Eq. (27) should range over all the roots of Eq. (32). However, from Eqs. (32) and (28), to 0^{th} order in ε, one sees that the squared-speed is

$$v^2_0 = \frac{2u_{n,0}}{\sin 2u_{n,0}} = \frac{u_{n,0}}{\sin u_{n,0} \cos u_{n,0}} = \frac{1}{\cos^2 u_{n,0}} = 1 + \tan^2 u_{n,0} = 1 + u^2_{n,0}. \tag{33}$$

Each root of Eq. (32) (for $n \neq 0$) therefore demands the trajectory have a specific superluminal speed. Since the speed is fixed, it must take the value of just one of the speeds indicated by Eq. (33). For this speed, to 0^{th} order in ε, Eq. (32) will have

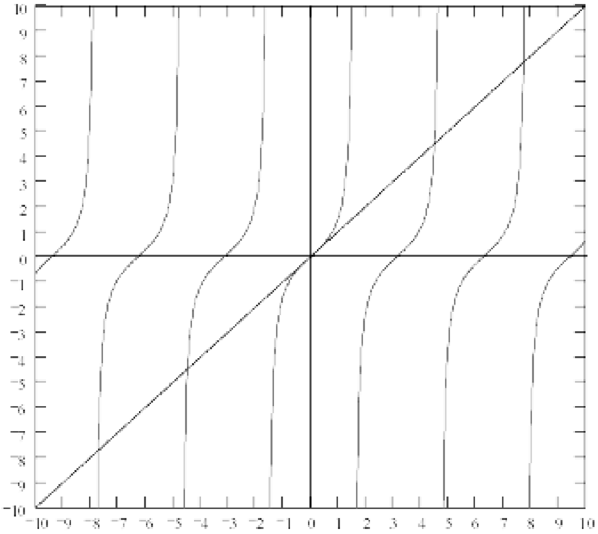

Figure 2: Graph showing $f(x) = tan(x)$ and $f(x) = x$ whose intersection points are the roots of Eq. (32).

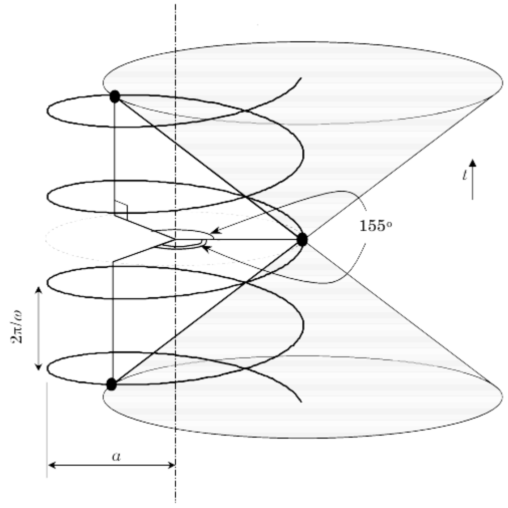

Figure 3: Circular motion as a space-time helix, showing three points in electromagnetic contact. The angles shown are for the $p = 1$ root of Eq. (32) - $u_1 = 4.49$ and the corresponding speed of $v_1 = 4.6$.

just three solutions: $u = 0$, and $u = \pm x$ where x is a particular non-zero solution of $\tan x = x$. (As noted above, the first of these is a double solution.) Let us say the trajectory is in the state $n = p$ for some particular, fixed, p. Then these roots give the contributions to Eq. (25)

$$I_p = e^2 \omega \int dt \left(K_0^+ + K_p \right), \tag{34}$$

and the requirement Eq. (27) is now that

$$\lim_{\varepsilon \to 0_+} \left(\varepsilon \left(K_0^+ + K_p \right) \right) = 0. \tag{35}$$

Since p is a fixed number, the speed in Eq. (15) can be labeled accordingly,

$$v^2{}_p = \sum_{k=0} v^2{}_{p,k} \varepsilon^k \tag{36}$$

and Eq. (33) becomes

$$v^2{}_{p,0} = 1 + u^2_{p,0}. \tag{37}$$

The first few roots of Eq. (32) and corresponding speeds are given in Table 1.

p	0	1	2	3	4	5	...	large p
u_p	0	4.49	7.73	10.90	14.07	17.22		$\left(p + \frac{1}{2}\right)\pi$
v_p	1	4.60	7.79	10.95	14.10	17.25		$\left(p + \frac{1}{2}\right)\pi$
θ_p	0	515	885	1250	1612	1973		$(2p + 1)\, 180$
cycles	0	1.43	2.46	3.47	4.48	5.48		$p + \frac{1}{2}$

Table 1

Roots of Eq. (32) and the corresponding speeds given by Eq. (37). θ_p is the angle in degrees between adjacent points of electromagnetic contact. For large p the corresponding number of cycles of the helical trajectory approaches $p + \frac{1}{2}$.

As stated above, the number of roots solving Eq. (32) depends on the magnitude of the speed. The additional requirement that for $p \neq 0$ the K_p diverge like $1/\varepsilon$ as $\varepsilon \to 0$ is the requirement that when the trajectory intersects its own light cone at $\mathbf{x}'$ emanating from $\mathbf{x}$ say, it does so in such a manner that the projection of its velocity in the direction $\mathbf{x} - \mathbf{x}'$ is the speed of light. That is, the trajectory grazes its own light cone at a tangent. It is this additional tangency requirement, combined with the light-cone condition, Eq. (12), that gives rise to a discrete set of allowed velocities as in Table 1. Fig. 3 illustrates the particular case $p = 1$.

Given the particular state of the trajectory p, it will be possible to express the quantities $u_{0,k}$, $u_{p,k}$, $k \neq 0$, and $v^2{}_{p,k}$, $k \neq 0$ all in terms of $u_{p,0}$. For example, recalling Eq. (21) and using Eq. (33)

$$u_{0,1} = 1/u_{p,0}. \tag{38}$$

2.5. *Condition for cancellation of the singularities*

Given Eq. (28) , the second of Eq. (19) requires

$$v^2_{p,1}\left(c_p-1\right)=0\Rightarrow v^2_{p,1}=0, \tag{39}$$

($c_p \neq 1$ because $u_{p,0}$ is neither 0 nor 2π). With this, K_p from Eq. (29) is

$$K_p=\frac{v^2_{p,0}c_p-1}{2\varepsilon u_{p,1}\left|v^2_{p,0}c_p-1\right|}+O\left(\varepsilon^0\right)=\frac{\operatorname{sgn}\left(v^2_{p,0}c_p-1\right)}{2\varepsilon u_{p,1}}+O\left(\varepsilon^0\right). \tag{40}$$

But

$$c_p=\cos 2u_{p,0}=\frac{1-\tan^2 u_{p,0}}{1+\tan^2 u_{p,0}}=\frac{1-u^2_{p,0}}{1+u^2_{p,0}}, \tag{41}$$

which with Eq. (33) gives

$$v^2_{p,0}c_p-1=-u^2_{p,0}, \tag{42}$$

which is always negative. Therefore Eq. (40) is

$$K_p=-\frac{1}{2\varepsilon u_{p,1}}+O\left(\varepsilon^0\right). \tag{43}$$

With Eqs. (26) and (43), implementation of the condition Eq. (35) requires that

$$u_{p,1}=u_{0,1}=1/u_{p,0}. \tag{44}$$

In summary, if x is the p^{th} root of $x=\tan x$, then

$$\begin{aligned} u_0 &= \quad \varepsilon/x+O\left(\varepsilon^2\right)\\ u_p &= \quad x+\varepsilon/x+O\left(\varepsilon^2\right)\\ v^2_p &= \quad 1+x^2+O\left(\varepsilon^2\right) \end{aligned} \tag{45}$$

gives an action that is finite over a finite interval as $\varepsilon \to 0$. Alternatively, the roots can be written in terms of the speed:

$$\begin{aligned} u_0 &= \varepsilon/\sqrt{v^2_p-1}+O\left(\varepsilon^2\right)\\ u_p &= \sqrt{v^2_p-1}+\varepsilon/\sqrt{v^2_p-1}+O\left(\varepsilon^2\right) \end{aligned} \tag{46}$$

where, from Eqs. (32) and (33) v_p is the p^{th} root of

$$\tan\sqrt{v^2_p-1}=\sqrt{v^2_p-1}. \tag{47}$$

3. Discussion and speculation

3.1. *Description of the motion*

The motion Eq. (5) renders the action Eq. (2) finite over a finite time interval provided the conditions Eqs. (46) and (47) are satisfied. Consequently it is conjectured that these 'special' motions are a solution of the Euler equations (though see below), and therefore are physical possibilities in CEM.

In Fig. 3 is shown, for the case $p = 1$ the three points of electromagnetic contact corresponding to the double root $n = 0$ and the roots $n = \pm 1$. The three point structure rotates with the helix forming a continuous space-time surface of connections.

It is plausible that at spatial distances from the helix large compared to a, and averaged over times large compared to the period $2\pi/\omega$, the electric field will not average to zero, and will fall off as $1/r^2$. Assuming this to be the case, then under those conditions the space-time helix will look like a static classical charge. After averaging, the magnitude of the charge sourcing the field will be some dimensionless numerical factor times the e employed in Eq. (2) – the factor to be determined from a careful analysis of interaction with *external* charges.

Again, at spatial distances from the helix large compared to a, and averaged over times large compared to the period, the circular motion will give rise to a magnetic dipole moment. From purely dimensional arguments, after averaging, the magnitude of the dipole moment must be some dimensionless numerical factor times e/ω.

In the event that the helix is approached by a (now fictitious) classical test charge, at any t-time, there will be up to three points of interaction. These three are: the collision point with the space-time helix, and the two points of light-cone contact of that point and past and future turns on the helix - the three points shown in Fig. 3. That is, there will appear to be up to three scattering centers for the approaching test charge.

3.2. *Total energy*

It has been demonstrated that the action is finite or zero for the considered class of motions only for discrete values of parameters of the motion. Other parameter values fail to cancel the Coulomb term. For those un-canceled motions the action is singular, for which it is safe to assume that the Coulomb energy continues to dominate the total, and which, therefore, remains positive infinite.

However, though it seems likely that the action is an extremum for the proposed special motions satisfying Eq. (46) and Eq. (47), it must be noted that it this not yet been demonstrated, even though other parametrically 'nearby' actions are singular.

To demonstrate the action is an extremum it is necessary to show that the proposed special motions solve the Euler equation for Eq. (2)

$$F^{ab}(\lambda)\, u_a(\lambda) = 0 \tag{48}$$

where F is the anti-symmetric field strength tensor derived from the trajectory itself, $x(\lambda)$. I.E., the motion must be such that the total self-force is zero [3]. If that turns out to be the case, then the total energy system is zero [16].

3.3. *Mass and stability*

Where does mass come from? From the above it seems likely that the candidate solution has zero mass. If the trajectory is to form the basis of a unified description of mass and gravity then this, perhaps, is good news. In such a theory it is safe to say that mass and gravity must both come from electromagnetic interaction, and therefore the length scale associated with mass must come from the distribution of environmental matter (more exactly, the two length scales are related). Beyond that the question remains unanswered - somehow inertial mass must emerge as a more-or-less localized positive energy of interaction. The issue of mass seems not unrelated to that of stability. How stable is the trajectory under the influence of external forces? In the framework of the model presented here, this question must be answered by considering the stability of two trajectories each with helical motion as they are allowed to act upon each other.

3.4. *Is there any connection with real physics?*

Is there any connection between self-interacting electromagnetic motion producing a self-canceling force, and any of the fundamental particles? In particular, is there any connection between the model space-time helix discussed above and its excited states corresponding to the different roots of Eq. (32), and the proton, and other Baryons?

It seems important that cancellation of the Coulomb energy is possible only if the trajectory moves at significantly superluminal speeds (at least $v = 4.6$ for circular motion). It is hard to see how this conclusion could be challenged by consideration of other motions. Yet the minimum requirement for cancellation of the Coulomb term is only that the cancellation-causing part of the trajectory travel at light speed - significantly less than the speed of the self-interacting circular trajectory. A distinct alternative possibility for a non-singular trajectory then, is that cancellation is a cooperative effort involving other, external, trajectories traveling much closer (but $\geq$) to the speed of light. There seems no a priori reason that these other trajectories need be local to each other. If so, then there would appear to be two distinct classes of trajectory: Those that are non-singular through local self-interaction of a superluminal trajectory, and those that are non-singular through non-local cooperative interaction of close-to-light-speed trajectories. Could there be a correspondence between this classification of alternatives, and the distinct families of Hadrons and Leptons? Obviously much work remains to be done to answer these questions. Since this investigation is in very incomplete, the reader is left to draw his or her own conclusions on the probability that these self-canceling trajectories correspond to anything physical.

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