Memorandum

To: Record

From: Hal Puthoff,

Date: December 31, 2007

Re: Gravitoelectric and Gravitomagnetic Forces

The transition from (weak field) Einstein GR equations to EM-like Maxwell equations for gravitoelectric and gravitomagnetic forces is not without confusion in the literature, due primarily to the appearance of sign changes and factors of 2 and 4, depending on the definitions chosen to transform EM to GR variables [1,2]. Different researchers choose to handle these issues in different ways, ranging from leaving the (Lorentz-like) force law intact in form and instead modifying the Maxwell-like field equations [3,4], through leaving the Maxwell-like equations intact and modifying the force law [5,6], to keeping both intact but defining parameters in such a manner that the expression for the velocity of propagation of gravity waves has the (incorrect) value of c/2 instead of c [7], and, finally, to ignoring the factors altogether [8]. Here we summarize the significant issues with an eye toward transparency.

Perhaps the clearest pedagogical treatment can be found in References [3,4]. One begins with Maxwell's equations in the form (MKS units):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_0}, \ \nabla \bullet \mathbf{B} = 0, \ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Introduction of the potentials $(\varphi, \mathbf{A})$ by $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$, $\mathbf{B} = \nabla \times \mathbf{A}$, related by the Lorentz condition $\nabla \bullet \mathbf{A} = -\frac{1}{c^2}\frac{\partial \varphi}{\partial t}$, leads to

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}, \ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J},$$

where ε_0 and μ_0 are the permittivity and permeability of the vacuum. To determine the force on a charged particle *q*, the above is augmented by the Lorentz force equation

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}\,.$$

It is then relatively straightforward[1] to show that, for slowly moving particles in weak gravitational fields, Einstein's equations can be recast into a (modified) Maxwell-like set of equations analogous to the above, called Gravitoelectromagnetic (GEM) equations, of the form [3,4]

$$\nabla\times\mathbf{E}_g = -\frac{\partial\left(\mathbf{B}_g/2\right)}{\partial t}, \quad \nabla\bullet\mathbf{E}_g = 4\pi G\rho\,, \;\; \nabla\bullet\left(\mathbf{B}_g/2\right) = 0\,, \;\; \nabla\times\left(\mathbf{B}_g/2\right) = \frac{4\pi G}{c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}_g}{\partial t}\,,$$

$$\mathbf{E}_g = -\nabla\varphi_g - \frac{\partial\left(\mathbf{A}_g/2\right)}{\partial t}\,, \left(\mathbf{B}_g/2\right) = \nabla\times\left(\mathbf{A}_g/2\right), \;\; \nabla\bullet\left(\mathbf{A}_g/2\right) = -\frac{1}{c^2}\frac{\partial\varphi}{\partial t}\,,$$

$$\nabla^2\varphi_g - \frac{1}{c^2}\frac{\partial^2\varphi_g}{\partial t^2} = -4\pi G\rho\,, \;\; \nabla^2\left(\mathbf{A}_g/2\right) - \frac{1}{c^2}\frac{\partial^2\left(\mathbf{A}_g/2\right)}{\partial t^2} = -\frac{4\pi G}{c^2}\mathbf{J}\,,$$

where the vacuum parameters ε_0 , μ_0 are replaced by $\varepsilon_0 \to 1/4\pi G$, $\mu_0 \to 4\pi G/c^2$; ρ and $\mathbf{J}$ are now mass and mass-current densities (assumed positive as sources); and the gravitomagnetic field variables $\mathbf{B}_g$, $\mathbf{A}_g$ appear halved in comparison with their EM counterparts due to the fact that gravitomagnetic field generation by mass currents is twice as efficient as the gravitoelectric field generation by mass charges.[2] Once the above equations have been solved as in their counterpart EM cases, for a slowly moving test particle in the stationary GEM field of a source ($\partial\mathbf{A}_g/\partial t = 0$, $\partial\varphi_g/\partial t = 0$) the factor of 2 enters yet again into the gravitomagnetic part of the gravitational Lorentz force equation

$$\mathbf{F}_g = q_E\mathbf{E}_g + q_B\mathbf{v}\times\mathbf{B}_g$$

due to the assignment $q_E \to -m$, $q_B \to -2m$.[3] This expresses once again the 2:1 ratio of gravitomagnetic to gravitoelectric charge effects, both for the motion of test particles and for ideal test gyroscopes. A consequence of this feature is that the gravitomagnetic part of the Lorentz force, $q_B\mathbf{v}\times\mathbf{B}_g$, is 4 times its corresponding EM value, given the factors of 2 in both the definition of q_B and in the calculation of $\mathbf{B}_g$ from its sources.[4]

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[1] See Appendix for an outline of the steps.

[2] This additional factor of 2 in the gravitomagnetic term can be traced back to the spin-2 character of the linearized gravitational field, as contrasted with the spin-1 EM field for which the ratio of magnetic to electric effects due to electrical current and charge is unity.

[3] In comparison with the EM case, though mass charge and current sources ρ and $\mathbf{J}$ are assumed positive, minus signs for the test-particle response are due to the fact that, unlike electric charges, like "mass charges" attract.

[4] See Ref. [4] for generalization of the Lorentz force law to the case of higher-velocity test particles.

With the above relationships clarified in some detail, for force and torque calculations one now has the option, if one wishes, to redefine the gravitomagnetic variables $\mathbf{B}_g$, $\mathbf{A}_g$ so as to eliminate the factors of 2 in the GEM equations (i.e., let $\mathbf{B}_g/2 \rightarrow \mathbf{B}_g$, $\mathbf{A}_g/2 \rightarrow \mathbf{A}_g$ so as to reestablish the original form of Maxwell's equations) and, instead, incorporate the net effect of signs and numerical factors into the Lorentz force equation explicitly,

$$\mathbf{F}_g = -m\mathbf{E}_g - 4m\mathbf{v} \times \mathbf{B}_g \ ,$$

an option exercised in many treatments [1,5,6].

REFERENCES

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APPENDIX

For weak gravitational fields the spacetime metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\upsilon}$ is the Minkowski metric tensor (-1, 1, 1, 1) and $h_{\mu\nu}$ is a first-order perturbation. Definition of potentials $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, where $h = tr\left(h_{\mu\nu}\right)$, along with use of a "Lorentz" gauge condition $\bar{h}^{\mu\nu}{}_{,\nu} = 0$, permits Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

to take the form of a set of gravitational field equations

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} .$$

Analogous to the EM case, the above equation possesses (nonhomogeneous) solutions of the form

$$\bar{h}_{\mu\nu} = \frac{4G}{c^2}\int \frac{T_{\mu\nu}\left(ct - \left|\mathbf{x} - \mathbf{x}'\right|, \mathbf{x}'\right)}{\left|\mathbf{x} - \mathbf{x}'\right|}d^3x' .$$

With the choice of source terms $T^{00} = \rho c^2$ and $T^{0i} = cj^i = c\rho v^i$ for the gravitational charge and current densities, respectively, the introduction of the gravitoelectric scalar and gravitomagnetic vector potentials $\bar{h}_{00} = \frac{4\varphi_g}{c^2}$, $\bar{h}_{0i} = -\frac{2A_i}{c}$, along with ignoring terms such as $\bar{h}_{ij} = O\left(c^{-4}\right)$ and lower, the spacetime metric is given by

$$ds^2 = -\left(1 - \frac{2\varphi_g}{c^2}\right)c^2dt^2 - \frac{4}{c}\left(\mathbf{A}_g \bullet d\mathbf{x}\right)dt + \left(1 + \frac{2\varphi_g}{c^2}\right)\delta_{ij}dx^i dx^j ,$$

and the GEM equations follow.

The geodesic equation of motion of a test particle with proper time τ and four-velocity $u^\mu = dx^\mu / d\tau$

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma}u^\rho u^\sigma = 0$$

can then, for a slowly moving particle in a stationary GEM field, be written in a form analogous to the Lorentz force law as given in the main body of the text. Modification of the Lorentz force law for higher speed particle motion in a GEM field is given in Ref. [4].