Frame dragging and electromagnetic plane wave phase

Working Notes

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February 16, 2008

1 Results

We suppose that an EM plane wave is propagating in the $+x$ direction, with angular frequency ω_0, and with unperturbed phase $\omega_0(t-x)$. We now introduce a spinning mass located at the origin, with an angular momentum vector $\boldsymbol{L}$ that points in the positive z axis direction. The result I've found is that at first order in all small quantities, the phase of the plane wave gets altered to

$$\phi(t,x,y,z) = \omega_0(t-x) - \frac{2xy\,\omega_0\,G\,|\boldsymbol{L}|}{c^4\sqrt{x^2+y^2+z^2}\,(y^2+z^2)} \tag{1}$$

where G is Newton's constant, and c the speed of light. (Note that this is just for the 'magnetic' part of the gravito-electromagnetic field — I've not included the direct attraction part.)

Basically, for the case where the plane of the ring laser or FOG is perpendicular to the z axis, which we shall assume, then all the results we need are available in this one formula. In particular, then due to rotational symmetry about the z axis it doesn't matter that we chose the x-direction for the wave.

As an example, and to check how this result relates to using time delay along geodesics, we consider a rectangular round trip as follows.

1. From $(x,y) = (-l/2,-l/2) \mapsto (l/2,-l/2)$
2. Followed by $(l/2,-l/2) \mapsto (l/2,l/2)$
3. Followed by $(l/2,l/2) \mapsto (-l/2,l/2)$
4. Followed by $(-l/2,l/2) \mapsto (-l/2,-l/2)$

all carried out at some constant $z = z_0$. One beam in the interferometer is taken to traverse this path in the given direction ((1) to (4)), and the other counter-propagating beam performs (4) to (1) with each leg reversed. Note we are going to ignore whatever phase change takes place at each of the corner reflection points, since presumably these will be the same for both beams.

Due to the rotational symmetry, we just need to work out the phase change on segment 1, and then the rest will be the same.

On segment 1, equation (1) gives a phase change (due to the spinning mass) of

$$\frac{8\,l^2\,\omega_0\,G\,|\boldsymbol{L}|}{\sqrt{2\,l^2+4\,{z_0}^2}\,(l^2+4\,{z_0}^2)\,c^4} \tag{2}$$

The total phase change, on returning to the starting point for the first beam is therefore 4 times this.

For the counter-propagating beam, the phase change will be minus what we have just obtained. (One can get this from equation (1) using the rotational symmetry arguments already given, plus seeing that we just need to evaluate the end points in the opposite order.)

The overall phase difference that will be measured in the interferometer for this case is therefore

$$\Delta\phi_{\text{total}} = \frac{64\,l^2\,\omega_0\,G\,|\boldsymbol{L}|}{\sqrt{2\,l^2+4\,{z_0}^2}\,(l^2+4\,{z_0}^2)\,c^4} \tag{3}$$

I've so far checked this against the geodesic time delay results for the special case of the $z_0 = 0$ plane. (Note this is how the Canterbury ring laser is set up with respect to the rotating superconductor — though of course offset paths in the $z_0 = 0$ plane are needed compared to the symmetrically placed square above — this is easy to carry through.)

What one finds using geodesic paths, is that for propagation in the plus x direction (and with $z = 0$), then the apparent coordinate velocity of a photon is

$$\frac{dx}{dt} = c - \frac{2\,G\,|\boldsymbol{L}|\,y}{c^2(x^2+y^2)^{3/2}} \tag{4}$$

Treating the perturbation as small, and then inverting and integrating, we obtain

$$c\,t(x) = x\left(1 + \frac{2\,G\,|\boldsymbol{L}|}{c^3\sqrt{x^2+y^2}\,y}\right) + \text{const.} \tag{5}$$

Thus for the photon to go from $x = -l/2$ to $x = +l/2$ at a general y takes a time

$$\frac{l}{c}\left(1 + \frac{4\,G\,|\boldsymbol{L}|}{c^3\sqrt{l^2+4y^2}\,y}\right) \tag{6}$$

Considering a complete round-trip path as above, and comparing with the counter-propagating beam, we get a time difference between the two trips of

$$\Delta t_{\text{total}} = \frac{32\sqrt{2}\,G\,|\boldsymbol{L}|}{c^4\,l} \tag{7}$$

and hence a phase difference

$$\Delta\phi_{\rm total} = \frac{32\sqrt{2}\,\omega_0\,G\,|\boldsymbol{L}|}{c^4\,l} \tag{8}$$

which agrees with equation (3) evaluated at $z_0 = 0$.

We should perhaps note that equation (1) leads to an apparent problem if we are in the plane $z = 0$ and have $y = 0$ also, since then the phase correction term blows up. Of course in this case we are effectively considering a photon that is going to intersect with the central mass, where the frame-dragging effect blows up anyway, but it is perhaps surprising that we cannot use the phase even along a section of the path that could be well away from the origin. This needs further looking at, but my belief is that (1) is nevertheless valid everywhere that the correction (i.e. second) term is small, which certainly applies to all the cases considered above, and likely to be encountered in practice.

2 Using the phase formula in practice

The result (1) contains all that we need, but is not in a very convenient form for application to more complicated paths than we have considered above. In particular, while rotational symmetry enabled us to argue that the phase shift along each leg had to be the same, we could only do this because of the $\pi/2$ rotational symmetry enjoyed by a square.

To be able to work with more complicated paths, we first generalise (1) to general angles of propagation in the $x-y$ plane. This can be carried out by writing

$$(x, y) = (r\cos\phi\sin\theta, r\sin\phi\sin\theta) \tag{9}$$

changing the azimuthal angle ϕ to $\phi - \alpha$ and then resubstituting for x and y. The overall result is that the new general phase term is

$$\begin{aligned}\phi(t,x,y,z) = \omega_0(t - (x\cos\alpha + y\sin\alpha))\\ -\frac{(2\,yx\cos(2\,\alpha) - x^2\sin(2\,\alpha) + y^2\sin(2\,\alpha))\,2\omega_0\;G\,|\boldsymbol{L}|}{\sqrt{x^2+y^2+z^2}\,(2\,z^2 - x^2\cos(2\,\alpha) + x^2 + y^2\cos(2\,\alpha) + y^2 - 2\,xy\sin(2\,\alpha))}\end{aligned} \tag{10}$$

and that this is appropriate for propagation at azimuthal angle α with respect to the x-axis. (Note we have put $c = 1$ in this — factors of c will come back at the end.)

Now, adopting a photon point of view, the vector $\boldsymbol{v} = (\cos\alpha, \sin\alpha, 0)$ is the spatial velocity of the photon. To work out the phase change along a path, we clearly need to integrate $\boldsymbol{v}\cdot\boldsymbol{\nabla}\phi$ along the path. At this point, a nice result emerges.

Carrying out $\boldsymbol{v}\cdot\boldsymbol{\nabla}\phi$ with $\boldsymbol{v} = (\cos\alpha, \sin\alpha, 0)$ on the gravity-induced part of ϕ above, we find (with no approximation) that

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\phi = \frac{2}{r^3}(x\sin\alpha - y\cos\alpha)\omega_0 G\,|\boldsymbol{L}| \tag{11}$$

where $r = \sqrt{x^2+y^2+z^2}$. This is clearly a big simplification! It shows us that the orbital angular momentum of the photon couples directly to the angular momentum of the source.

So suppose we have a closed path in space parameterised by λ — we write this as $\boldsymbol{x}(\lambda)$. The velocity $d\boldsymbol{x}/d\lambda$ will be in the direction of $\boldsymbol{v}$, so it is easy to see that an explicit formula for phase change around the path is

$$\Delta\phi = -2\omega_0 G\,\boldsymbol{L}\cdot\left(\oint \frac{1}{|\boldsymbol{x}(\lambda)|^3}\left[\boldsymbol{x}(\lambda)\wedge\frac{d\boldsymbol{x}}{d\lambda}\right]d\lambda\right) \tag{12}$$

The total phase difference between this path and a counterpropagating one will then be twice this. I.e., putting back the factors of c

$$\Delta\phi_{\text{total}} = -\frac{4\omega_0 G}{c^4}\boldsymbol{L}\cdot\left(\oint \frac{1}{|\boldsymbol{x}(\lambda)|^3}\left[\boldsymbol{x}(\lambda)\wedge\frac{d\boldsymbol{x}}{d\lambda}\right]d\lambda\right) \tag{13}$$

(and for a FOG this will be multiplied up by however many turns there are).

Some comments on this formula:

1. The result is written in geometric algebra form, i.e. both $\boldsymbol{L}$ and $\boldsymbol{x}(\lambda)\wedge\frac{d\boldsymbol{x}}{d\lambda}$ here are *bivectors*. This is obviously not crucial, and one could quickly turn it into a more standard vector form, but I'd need to be careful about signs, and need longer to do this. Up to sign, it can be taken that we are just dotting together the angular momentum vectors of the source and photon (the latter in the sense of its orbital angular momentum).

2. The formula has been written for a general path, not restricted to a plane perpendicular to $\boldsymbol{L}$. I think the result is probably true in this general sense, but please note it's only been proved here for the more restrictive case.

3. If we ignore the factor $|\boldsymbol{x}(\lambda)|^3$ in the denominator of the integral, then we see that the result of the integral is the vector area traced out by the path. This is exactly what we'd expect in the pure rotation (Sagnac) case. So the eventual difference with that case, is to modulate the vector area integral with the reciprocal distance cubed to the source.

3 Evaluation of the phase difference in some specific cases

1. For a square of side l lying in the $x - y$ plane and with centre at $(0, 0, z_0)$ (i.e. the case studied above), then (13) reproduces the result already found in (3), so this is a good check.

2. If we replace this square with a circle, of radius R, we get

$$\Delta\phi_{\rm total}(circle) = \frac{8\,G\,\omega_0\,|\boldsymbol{L}|\,\pi R^2}{c^4(z_0^2 + R^2)^{3/2}} \tag{14}$$

3. If we take an offset square, of side length l lying in the $z = 0$ plane, and to simulate the Canterbury RLG setup having its centre at $(x, y) = (l/2 + \delta, 0)$, where $\delta \ll l$, then, to first order in δ we get

$$\Delta\phi_{\rm total}(Canterbury) = -\frac{8\,G\,\omega_0\,|\boldsymbol{L}|}{c^4\delta} + \frac{8\sqrt{5}\,G\,\omega_0\,|\boldsymbol{L}|}{c^4 l} \tag{15}$$

   This is clearly dominated by the section of path close to the spinning mass, with the rest of the RLG having little effect on the answer. Note the leading term is *negative* compared to the preceeding cases. (The sense taken for the paths was the same as before, i.e. anticlockise viewed from above for the first path, clockwise for the counterpropagating one.)

4. As a numerical comparison, let's assume that a FOG in the Martin Tajmar apparatus has radius 25 mm and is 45 mm above the spinning disc (I'm not sure of the actual dimensions!). We can presumably ignore the number of turns since this will be taken out in the calibration as regards rotation rate returned.

   For the Canterbury experiment, we will take the 'square' to have side length 30 m with the spinning mass at an offset of 195 mm from the centre of one side.

   Then we get:

$$\begin{aligned} \Delta\phi_{\rm total}(Canterbury) &\approx -40\,\omega_0\,G\,|\boldsymbol{L}|/c^4 \\ \Delta\phi_{\rm total}(Tajmar) &\approx +115\,\omega_0\,G\,|\boldsymbol{L}|/c^4 \end{aligned} \tag{16}$$

   Although the dimensions for Martin's setup and FOGs will be wrong, the above is a possible illustration of the type of ratios involved.

4 Evaluation for offset circles

In Martin's apparatus, the FOGs are not on the rotation axes, but offset from it, so we would like a formula for the phase change around a circle lying perpendicular to the z-axis, with an arbitrary centre in space.

We suppose the radius of the FOG loop is R (as above), and that the centre is at radial distance ρ out perpendicularly from the z axis, and at height z_0.

The answer is

$$\Delta\phi_{\text{total}}(\text{offset circle}) = \frac{8G\omega_0|\boldsymbol{L}|}{c^4\sqrt{z_0^2+(\rho-R)^2}} \times \left[K\left(\frac{2i\sqrt{\rho R}}{\sqrt{z_0^2+(\rho-R)^2}}\right) - \left(\frac{\rho^2+z_0^2-R^2}{z_0^2+(\rho+R)^2}\right) E\left(\frac{2i\sqrt{\rho R}}{\sqrt{z_0^2+(\rho-R)^2}}\right)\right] \tag{17}$$

where K and E are complete elliptic integrals of the first and second kinds repectively.

(Note the complete elliptic integrals are defined in terms of the *square* of their argument, so there is no problem about choice of branch in the result, which is always real.)

As a check on this result, we see that it correctly reduces to the on-axis result (14) when $\rho = 0$, since both $E(0)$ and $K(0)$ equal $\pi/2$.

This formula allows us to explore in detail the sensitivity to position of the expected result from a FOG. We illustrate this with two plots.

The first plot (Fig. 1), shows the total phase difference, normalised by dividing by $(8G\omega_0|\boldsymbol{L}|/c^4)$, as a function of vertical position along a line going through the centres of the FOGs LG3 ('above'), LG2 ('middle') and LG1 ('reference'), as depicted in Figure 5 of Tajmar et al. (*'Search for frame-dragging in the vicinity of spinning superconductors'*). The measurements used are as given in that paper, so z_0 ranges over 45.8 to 220.8 mm, and $\rho = 53.75$ mm. The other dimension we need is the radius of a FOG loop, which I have guessed from the external dimensions to be 25 mm. We can see that the signal, after an initial rise, drops to about 1/6th of its initial value (which is at LG3) by the time one reaches the reference gyro (LG1). If one takes the cubed ratio of the distances of LG1 and LG3 from the origin, one obtains 1/30th as the expected decrease, if this were the only effect, so the actual decrease is much less than this.

We can get a good idea of what is going on from the next figure, Fig. 2.

This shows the radial variations (i.e. horizontal cuts) at three different z_0's, corresponding to the heights of the LG1, LG2 and LG3 gyros.

One can see that the effect is maximum on the z-axis, but then declines away from there, and at a certain point changes sign. The actual radial distances of all three sensors are $\rho = 53.7$ mm, and one can see from the plot that the expected outputs

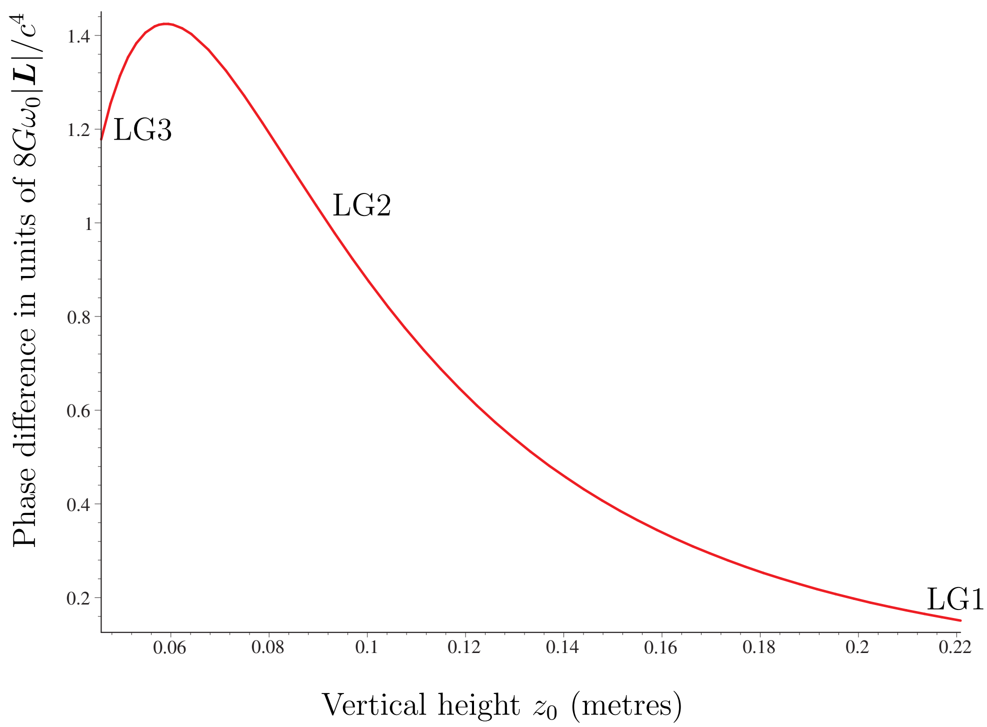

Figure 1: Variation of predicted total phase difference along a vertical line through the three FOGs LG3, LG2 and LG1 as shown in Fig. 5 of Tajmar et al.

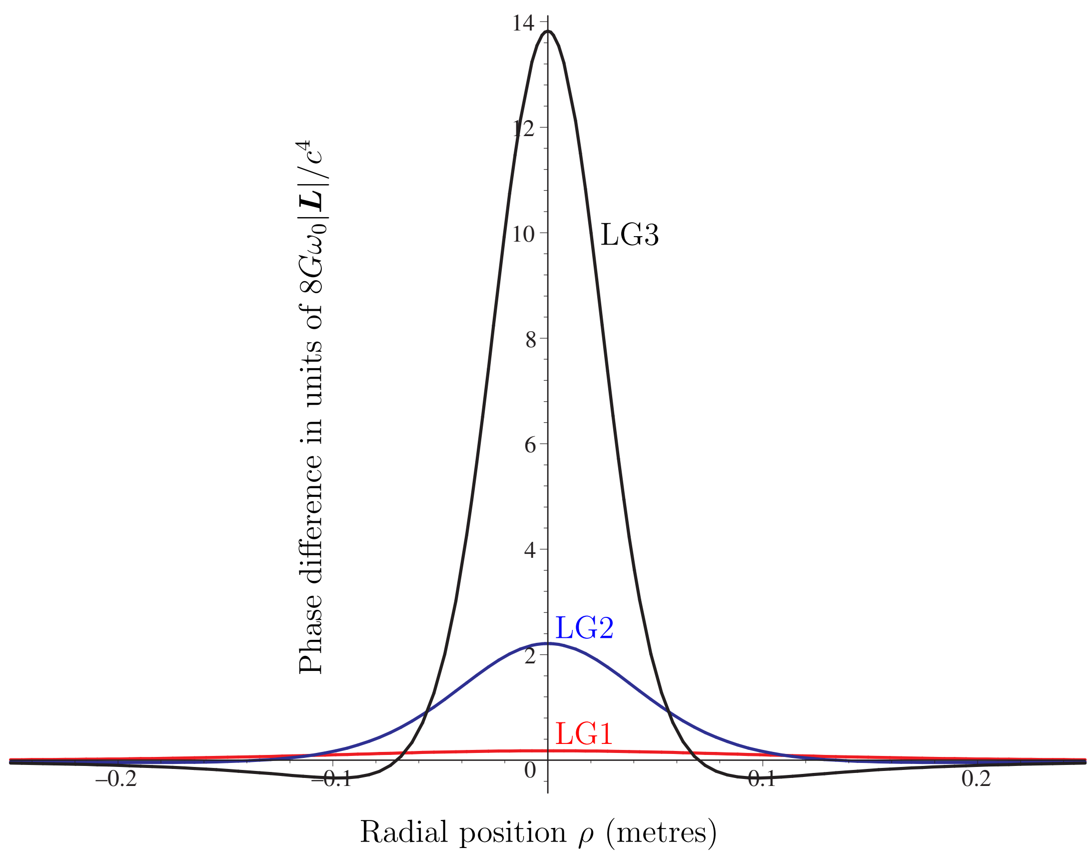

Figure 2: Variation of predicted total phase difference along three radial lines perpendicular to the z-axis. The red curve is for a radial line at the height (z_0) of the LG1 ('reference') sensor (220.8 mm); blue is at the height of the LG2 ('middle') sensor (92.8 mm); and black is at the height of the LG3 ('above') sensor (45.8 mm). The actual radial distances of these three sensors are all 53.7 mm.

from LG3 (the nearest to the spinning ring) and LG2 (the 'middle' position) are declining markedly there. This is what brings the ratio of outputs from LG3 to LG1 at this radial position down much lower than the 30 expected on a crude analysis, to the approx. 6 seen in Fig. 1.

It is clear that the behaviour of the phase difference in this frame-dragging case is very different to what one would get by taking a pure rotation (Sagnac) case, and weighting this afterwards with a $1/r^3$ factor. Performing the latter, will not produce the change of sign that occurs here, which is due to the $1/r^3$ factor acting differentially along a path. I.e. the factor must be put in before doing the integral, rather than using the Sagnac result, and then reweighting afterwards. (The latter appears to be what Graham et al. do, as described in Section 4 of their paper (*'Experiment to detect frame-dragging in a lead superconductor'*).)

Note however that the above results will depend sensitively on the radius of the fibre optic loop on the FOG, here taken as 25 mm. It is this which sets the scale at which the differential effects will be important.

A further health warning comes with the fact that while the phase calculations are likely to be very accurate for the assumed gravitomagnetic field, this field itself is only calculated in a certain approximation, and for the uses made here this may not be accurate enough. We discuss this now.

5 Accuracy of the computed gravitomagnetic field

The basic approximation involved in getting the gravitomagnetic field in the form we have used it, is that we take just the first two terms in the following expansion:

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} = \frac{1}{|\boldsymbol{y}|} + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{y}|^3} + \mathcal{O}\left(\frac{1}{|\boldsymbol{y}|^3}\right) \tag{18}$$

(see e.g. Hobson, Efstathiou & lasenby, p495). Here $\boldsymbol{x}$ is a 3d vector representing position within the source (i.e. here the spinning superconductor), and $\boldsymbol{y}$ is the field point, where we want to evaluate the gravitomagnetic field.

In this approximation, when the integral over the source is taken, then the only thing that is relevant about the source is its total angular momentum — the shape or size of the distribution of spinning material is irrelevant.

The approximation is valid, if the dimensions of the source are small compared to the distance to the field point. As a rough criterion, to get an error less than 20%, we could say that we need the distance to a field point to be 5 times the radius of a sphere encompassing the source.

In the current case (i.e. as described in the Tajmar et al. paper), the superconducting ring has radius 75 mm. This means that everything we have considered so far for this experiment is effectively in the near field, so it unlikely that our approximations will be very good. This therefore requires more detailed modelling to get proper answers for this case.